Spherically Symmetric Istantons of the Scale Invariant $SU(2)$ Gauged Grassmannian Model in $d = 4$

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Abstract

(Anti)self-dual solutions of the scale invariant $SO_{\pm}(4) \sim SU(2)$ gauged Grassmanian model are sought. A stronger (anti)selfduality condition for this system is defined, referred to as *strong self-duality*, and Spherically symmetric solutions of this *strong* (anti)self-duality equations are found in closed form. It is verified that these are the only solutions of the *strong* (anti)self-duality equations. The usual (anti)self-duality equations for the axially symmetric fields are derived and seen to be **not** overdetermined.

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1 Introduction

A hierarchy of scale invariant chiral $SO(4p)$ gauged Grassmannian models in $4p$ dimensions was introduced in Ref.[1], whose actions were minimised absolutely by a hierarchy of (anti)self-duality equations. For the particular case of spherically symmetric field configurations, these equations reduced to a single pair of coupled ordinary non-linear differential equations for all the members of this hierarchy given by equation (22) of Ref.[1], namely

$$k' = \mp\frac{2}{r}[k^2 + \frac{1}{2}(2k-1)(\cos f - 1)], \qquad f' \sin f = \pm\frac{4}{r}k(k-1). \tag{1}$$

(Note the misprint in the second member of Eq.(22) in Ref.[1]). The function $k(r)$ describes the spherically symmetric chiral $SO_{\pm}(2p)$ gauge fields

$$A_\mu = \frac{2}{r}[1-k(r)]\tilde{\Sigma}_{\mu\nu}\hat{x}_\nu, \tag{2}$$

while the function $f(r)$ describes the radially symmetric Grassmannian field $z^A{}_i = (z^a{}_i, z^\alpha{}_i)$, with $a, \alpha, i = 1, 2, .., 2^{2p-1}$,

$$z^a{}_i = \sin\frac{f(r)}{2}\,\delta^a{}_i, \qquad z^\alpha{}_i = \cos\frac{f(r)}{2}\,\hat{x}_\mu(\Sigma_\mu)^\alpha{}_i. \tag{3}$$

The spin matrices Σ_μ are chiral components of the gamma matrices in $4p$ dimensions and $\Sigma_{\mu\nu} = -\frac{1}{4}\Sigma_{[\mu}\tilde{\Sigma}_{\nu]}$ are the representations of the chiral $SO(4p)$, defined in Ref.[1]. The number of independent components of the complex $2^{2p-1} \times 2^{2p}$ field z, after deducting the $2^{2(2p-1)}$ real constraints due to $z^\dagger z = 1$ equals $3 \times 2^{2(2p-1)}$, from which must be subtracted further the number of components that can be fixed due to the inherent $SO_{\pm}(4p)$ gauge freedom of the Grassmanian dynamics, namely $2\lambda_p p(4p-1)$ with $\lambda_1 = \frac{1}{2}$ and $\lambda_p = 1$ for $p > 1$.

Equations (1) will be integrated, yielding the spherically symmetric solutions of the the whole hierarchy of self-duality equations of this system. Our focus however will go beyond the resticted case of spherically symmetry and we shall study the axially symmetric configurations of the self-duality equations in the $p = 1$ case, where our main objective will be to verify that these are **not** overdetermined and hence that the spherically symmetric solutions found explicitly are not the only solutions of these systems.

In the present work we shall restrict our considerations to the four dimensional, namely $p = 1$ case, exclusively. The reason is convenience and obvious physical relevance. Before imposing this restriction, we note two similarities between the hierarchy of scale invariant (chiral) $SO_{\pm}(4p)$ gauged Grassmannian models[1] and the hierarchy of scale invariant (chiral) $SO_{\pm}(4p)$ Yang-Mills(GYM) models studied previously[2]. Not surprisingly in both cases, the spherically symmetric solutions satisfy the same ordinary differential equation(s) for all members of the $4p$ dimensional hierarchy. This is due to the scale invariance of both hierarchies. The other similarity is the overdetermined nature of the hierarchy of (anti)self-duality equations for the $p > 1$ members of both hierarchies. In the case of the GYM hierarchy, it was shown[3] that except in the $p = 1$ case, the only solutions were the axially (and hence also the spherically) symmetric ones. In that case, the number of algebra-valued equations was $\frac{(4p)!}{2(2p)!^2}$ which exceeded the number of components of A_μ, namely $(4p-1)$, except for $p = 1$ where these two numbers were equal. In the present case the number of equations is equal to $2\lambda_p p(4p-1)\frac{(4p)!}{(2p)!^2}$. The number of (gauge) independent components of A_μ are again equal to $2\lambda_p p(4p-1)^2$ and the corresponding number for z is $3 \times 2^{2(2p-1)} - 2\lambda_p p(4p-1)$. Again, the number of equations exceeds the number of fields except for $p = 1$, when these are equal. The question as to whether axially symmetric solutions for $p > 1$ in the Grassmannian hierarchy exist will not be considered here, and instead we shall concentrate on the $p = 1$ case where we expect there are axially symmetric and less symmetric solutions since the number of equations is matched exactly by the number of independent fields. To this end, we will derive the restriction of the self-duality equations due to axial symmetry, and show that these equations are **neither** overdetermined **nor** underdetermined, but will not solve them.

Restricting to 4 dimensions, we procede to identify the (anti)selfdual solutions of the $SU(2)$ gauged Garassmannian model as *instantons*. To this end, we note that the solutions which we shall find below satisfy asymptotically pure-gauge conditions as stated by equations (28) and (29) in Ref.[1]. Here we state the large r asymptotic conditions relevant to the interpretation of the instanton vacuum, more completely, by including both self-dual and antiself-dual solutions. These asymptotic values are stated, repectively, as

$$\lim_{r\to\infty} k(r) = 1 \qquad\qquad \lim_{r\to\infty} f(r) = \pi \tag{4}$$

$$\lim_{r\to\infty} k(r) = 0 \qquad\qquad \lim_{r\to\infty} f(r) = 0. \tag{5}$$

In the antiself-dual case pertaining to (5), the field A_μ given by (2) can be expressed as a pure gauge $g^{-1}\partial_\mu g$ with $g = \hat{x}_\mu \sigma_\mu$, while z given by (3) tends to a constatnt valued matrix. In the self-dual case pertaining to (4), the field A_μ is gauge equivalent to the former, antiself-dual, connection via the gauge trasformation g, while the field z tends to a matrix expressed in terms of the same gauge group element g. Thus as for the pure Yang-Mills case[4], the asymptotically pure-gauge fields can be made time independent by making the relevant gauge group element g time independent, which in turn can be acheived by fixing $A_0 = 0$, in the temporal gauge. It will be possible to give a more symmetric discussion of the self- and antiself-dual cases below, when we analyse the gauge freedom of our solutions.

The sphaleron field configuration of this model was briefly discussed in Ref.[1], but we do not go into the details of this solution, since we are not immediately concerned with applications here. We suffice by noting that the sphaleron analysis corresponding to the Weinberg-Salam model[5] can be systematically carried out starting from the Chern-Simons form pertaining to the present model, given by

$$\Omega_\mu = \varepsilon_{\mu\nu\rho\sigma} Tr[A_\nu (F_{\rho\sigma} - \frac{2}{3} A_\rho A_\sigma) + z^\dagger D_\nu z\, F_{\rho\sigma}]. \tag{6}$$

In Section **2**, we give the definition of *strong self-duality*.and describe

briefly the relation of our solutions to those of a hierarchy of Grassmannian sigma models[6]. In Section **3** the spherically symmetric solutions are given and seen to be *strongly* (anti)self-dual, and it is shown further that the these are the only *strongly* self-dual solutions. Section **4** is devoted to a discussion of our results, with its main thrust being the derivation of the axially symmetric restriction of the *usual* self-duality equations, with the aim of verifying that these equations are not overdetermined.

2 Strong self-duality

In the first Subsection we introduce the *strong* (anti) self-duality conditions of our model, while in the second we introduce a new hierarchy of Grassmannian models whose 'instantons' are very naturally related to the *strongly* self-dual fields.

2.1 Strong self-duality

The *usual* (anti)self-duality equations for the 4 dimensional, $p = 1$, member of the gauged Grassmanian hierarchy are

$$^*F_{\mu\nu} = \mp G_{\mu\nu}, \tag{7}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ is the curvature of the $SU(2)$ Yang-Mills connection A_μ and $G_{\mu\nu}$ is defined as the covariant curl

$$G_{\mu\nu} = D_{[\mu} z \, D_{\nu]} z, \tag{8}$$

where the square brackets $[\mu\nu]$ denote antisymmetrisation, and using the following definition of the covariant derivative of z,

$$D_\mu z = \partial_\mu z - z \, A_\mu. \tag{9}$$

Since the Lagrangian density

$$\mathcal{L} = Tr[F_{\mu\nu}^2 + G_{\mu\nu}^2] \tag{10}$$

is bounded from below by the topological charge density

$$\varrho = \varepsilon_{\mu\nu\rho\sigma} Tr F_{\mu\nu} G_{\rho\sigma} = \partial_\mu \Omega_\mu \tag{11}$$

in the notation of (6), it is clear that the (anti)self-duality equations (7) solve the Euler-Lagrange equations

$$(1 - zz^\dagger) D_\nu z \, D_\mu G_{\mu\nu} = 0 \tag{12}$$

$$D_\mu F_{\mu\nu} + z^\dagger D_\mu \, G_{\mu\nu} + G_{\mu\nu} \, D_\mu z^\dagger z = 0 \tag{13}$$

arising from the arbitrary variations of the fields z and A_μ respectively, and having taken into account the constraint $z^\dagger z = 1$. That (7) solve (12) and (13) can readily be verified by use of the Bianchi identities.

To introduce *strong* self-duality, we express the covariant curl $G_{\mu\nu}$ in terms of the composite connection

$$B_\mu = z^\dagger \partial_\mu z \tag{14}$$

as follows

$$G_{\mu\nu} = D_{[\mu} B_{\nu]} - [A_\mu, A_\nu], \tag{15}$$

from which it follows that setting

$$B_\mu = A_\mu \tag{16}$$

leads to

$$G_{\mu\nu} = F_{\mu\nu}. \tag{17}$$

The condition of *strong* self-duality follows from the imposition of (16), leading to (17), according to which the *usual* (anti)self-duality equations (7) take the form

$$F_{\mu\nu} = \mp \, {}^*F_{\mu\nu}, \qquad\qquad G_{\mu\nu} = \mp \, {}^*G_{\mu\nu}. \tag{18}$$

(18) are the two equivalent statements of the *strong* (anti)self-duality equations of the system (10).

The *strong* (anti)self-duality equations (18) happen to be the *usual* (anti)self-duality equations of an *ungauged* Grassmannian model in 4 dimensions which coincides with the $p = 1$ member of the hierarchy of Lagrangian densities

$$\mathcal{L}_p = Tr \, (F^{(B)}(2p))^2 = Tr \, (G^{(B)}(2p))^2, \tag{19}$$

in the notation of Refs.[1][2] in which we have constructed the $2p$ form field strengths in terms of the 2 forms denoted as $F = F^{(B)}$ and $G = G^{(B)}$, to emphasise that the these quantities pertain to the composite connection B_μ. The topological charge densities presenting the lower bound on these Lagrangian densities are

$$\begin{aligned} \varrho_2 n &= \frac{1}{(2n-2)!}\varepsilon_{\mu_1\mu_2...\mu_{2n-1}\mu_{2n}} G_{\mu_1\mu_2}...G_{\mu_{2n-1}\mu_{2n}} \\ &= \frac{1}{(2n-2)! \times 2^{2n}}\varepsilon_{\mu_1\mu_2....\mu_{2n-1}\mu_{2n}} F^{(B)}_{\mu_1\mu_2}...F^{(B)}_{\mu_{2n-1}\mu_{2n}}, \end{aligned} \tag{20}$$

which are the n-th Chern-Pontryagin densities of $F^{(B)} = dB + B \wedge B$ in (20).

The most succinct way of describing the hierarchy of Grassmannian models (19) in $4p$ dimensions is to adapt the corresponding definition of the scale invariant generalised Yang-Mills(GYM) systems[2] by replacing the curvature of the Yang-Mills connection $F^{(A)}$ in the latter with $F^{(B)}$. These coincide with the $4p$ dimensional subset of the Grassmannian hierarchy introduced in Ref.[6], in which a scale invariant hierarchy in $4p+2$ dimensions is also given.

3 Spherically symmetric solutions

Subject to the constraint

$$k(r) = \sin^2 \frac{f(r)}{2} \tag{21}$$

on the functions $f(r)$ and $k(r)$, the pair of (anti)self-duality equations (1) reduce to

$$\frac{dk}{dr} = \mp\frac{2}{r}k(1-k), \tag{22}$$

which is integrated immediately to yield

$$k(r) = \frac{a^2}{r^2+a^2}\ , \qquad\qquad k(r) = \frac{r^2}{r^2+a^2} \tag{23}$$

which are the *unit* topological charge, antiself-dual and self-dual solutions respectively. The arbitrary constant of integration a appears as a result os the

scale invariance of the hierarchy of systems[1], just as for the YM(hierarchy), and the connection A_μ pertaining to the solution (23) is the (anti)self-dual BPST[7] connection. These solutions manifestly satisfy the asymptotically pure-gauge conditions (5) and (4)necessary for the instanton interpretation. We restate the conditions more completely to include the asymptotic values at the origin as well. For the antiself-dual case they are

$$\begin{aligned} &\lim_{r\to 0} k(r) = 1 \qquad &&\lim_{r\to\infty} k(r) = 0 \\ &\lim_{r\to 0} f(r) = \pi \qquad &&\lim_{r\to\infty} f(r) = 0 \end{aligned} \tag{24}$$

and for the self-dual case

$$\begin{aligned} &\lim_{r\to 0} k(r) = 0 \qquad &&\lim_{r\to\infty} k(r) = 1 \\ &\lim_{r\to 0} f(r) = 0 \qquad &&\lim_{r\to\infty} f(r) = \pi \end{aligned} \tag{25}$$

The additional constraint (21) appears is a consequence of the *strong* (anti)self-duality of the solutions (23). That we find that the spherically symmetric solutions are *strongly* (anti)self-dual with the BPST[7] connection satisfying (18), is not at all surprising in the light of our comments in the previous Section, where we identified the *strong* self-duality equations (18) with the self-duality equations of the Grassmannian model of Ref.[6], which in turn were integrated in closed form when restricted to spherical symmetry.

Indeed, given the (generalised)Yang-Mills connection $A_\mu = B_{\mu\nu}$ in closed form, we would find the *strongly* self-dual solution to the gauged Grassmannian systems of Ref.[1] in closed form by solving the differential equation (14) for z. This is done easily for the spherically symmetric field configuration (2) and (3) using the Clifford-algebraic properties of the Σ_μ matrices, but not for less symmetric restrictions of the connection A_μ as in the ADHM construction[8].

We have verified that the only *strongly* self-dual solutions are the spherically symmetric ones (23), by considering the axially symmetric retriction of ${}^*F = F$, the first member of the *strong* self-duality equations (23). We

have done this in hyperbolic coordinates[9] related to the variables $s = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and $t = x_4$ according to

$$s + it = \tanh(\rho + i\tau). \tag{26}$$

This conclusion is consistent with what we learn from the ADHM construction, according to which the YM connection $A_\mu = B_\mu$ can be parametrised as the composite connection (14), in which the field z can be a 2×4 matrix only for index 1.

4 Discussion

We have given the the spherically symmetric *unit* charge instantons of the hierarchy of chiral $SO_\pm(4)$ gauged Grassmannian models introduced in Ref.[1]. It turns out that these field configurations satisfy a stronger version of the (anti)self-duality equations stated in Ref.[1] which we have referred to as *strong* self-duality. The connection fields A_μ correponding to this solution coincide with the (hierarchy of) BPST[7] connections. They also coincide with the composite connection of the $4p$ dimensional subset of the hierarchy of Grassmannian sigma models of Ref.[6].

A peculiarity of the *strongly* self-dual equations (18) turns out to be that they have only spherically symmetric solutions. It is therefore very important to see if less symmetric restrictions of the *usual* self-duality equations (7) have nontrivial solutions. The counting of equations and independent field components carried out in Section **1** suggests that this could be the case in 4 dimensions, but not in higher dimensional cases with $p > 1$. Nevertheless it is necessary, and instructive, to verify this fact explicitly in the $p = 1$ case.

We start with the axially symmetric restriction of the *strong* self-duality equation (18), in the coordinates $s = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and $t = x_4$, and choose to work with the second member ${}^*G^{(B)} = G^{(B)}$ of (18) which we shall regard as the self-duality equation for the 4 dimensional Grassmannian model given

by the second member of (19). We employ the following axially symmetric restriction for the field z

$$z^a{}_i = \sin\frac{f}{2}\,(\cos\frac{a}{2}+i\hat{n}.\vec{\sigma}\sin\frac{a}{2})^a{}_i, \qquad z^\alpha{}_i = \cos\frac{f}{2}\,(\cos\frac{b}{2}+i\hat{n}.\vec{\sigma}\sin\frac{b}{2})^\alpha{}_i, \tag{27}$$

where the functions $f(s,t)$, $a(s,t)$ and $b(s,t)$ each depend on the two coordinates s adn t, and where we have used the notation $\hat{n} = \frac{\vec{x}}{s}$ for the unit radius vector in the $\mathbb{R}_3$ subspace of $\mathbb{R}_4$.

Let us start by counting the number of equations and independent functions. (18) consist of 3 independent $SU(2)$ valued equations, i.e. 9 equations. According to our counting in Section **1**, for $p = 1$ there are *also* 9 independent components of z. On the basis of this count we might expect to find solutions to (18) without subjecting it to any symmetries and hence also to the axially symmetric restriction. However, substitutuing the Ansatz (27) into ${}^*G^{(B)} = G^{(B)}$, we find that only the difference $c = \frac{1}{2}(a-b)$ of the functions of the functions a and b in (27) are determined by the equations and hence the equations become overdetermined, consistent with our conclusions of the previous Section and in agreement with the ADHM construction.

Using the notation $y_\alpha = (s,t)$, $\partial_\alpha = (\frac{\partial}{\partial s}, \frac{\partial}{\partial t})$, and $g = \ln\tan\frac{f}{2}$, the axially symmetric restriction of ${}^*G^{(B)} = G^{(B)}$ reduces to

$$\partial_\alpha g = \varepsilon_{\alpha\beta}\partial_\beta c \tag{28}$$

$$(\partial_1 g\partial_2 c - \partial_2 g\partial_1 c) = \frac{1}{2s^2}(1-\cos 2c). \tag{29}$$

(28) can be read as Cauchy-Riemann equations. That the system of three equations (28) and (29) is overdetermined is obvious since they determine only two functions $g(y)$ and $c(y)$. Any solutions they have, which we know from above that exist, must be more symmetric restrictions of (28) and (29). This is the *strongly* self-dual spherically symmetric solution, namely the second member of (23). Precisely the same arguments apply to the *strongly* antiself-dual solution, namely the second member of (23). Restricting ourselves to the self-dual case at hand, this restriction is imposed most naturally

by imposing the vanishing of the function $a(s,t)=0$ and by requiring that the function $g=g(r)$ be a (4 dimensional) radial function, as well as further requiring that

$$\cos\frac{b}{2}=\frac{t}{r}\,,\qquad\qquad \sin\frac{b}{2}=\frac{s}{r}. \tag{30}$$

The result is that the three equations (28) and (29) reduce to the single equation

$$r\,\frac{df}{dr}=\sin f, \tag{31}$$

which yields the spherically symmetric solution in question.

Before proceeding, we remark that this restriction is unique in the sense that it would not have been possible to impose instead the vanishing of the function $b(s,t)=0$. This is because we have opted in Ref.[1] to gauge the Grassmannian field z with the chiral $SO_-(4)$ gauge field, namely the $SU_L(2)$. Had we opted, equally legitimately, with the gauging with $SO_+(4)\sim SU_R(2)$ then we would have had to impose this restriction, along with the analogue of (30)

$$\cos\frac{a}{2}=\frac{t}{r}\,,\qquad\qquad \sin\frac{a}{2}=-\frac{s}{r}. \tag{32}$$

Having encountered the situation that the axially symmetric restriction of the *strong* self-duality equations are overdetermined in contradiction to our expectations based on the naive counting of the number of equations and the number of independent fields prior to imposing *strong* self-duality, it becomes necessary to test our corresponding expectation that the axially symmetric restriction of the *usual* self-duality equations (7) are **not** also overdetermined.

In this case, the $SU(2)$ gauge connection A_μ is **different** from the composite connection B_μ, (14), and its axially symmetric[10] restriction is

$$A_i=\frac{i}{2}[\frac{\phi_1}{s}\,\sigma_i\;\;+\frac{1}{s^2}(A_1-\frac{\phi_1}{s})\,x_i\vec{x}.\vec{\sigma}\;\;+(\frac{\phi_2+1}{s^2})\,\varepsilon_{ijk}\sigma_j x_k] \tag{33}$$

$$A_4=\frac{i}{2s^2}A_2\vec{x}.\vec{\sigma} \tag{34}$$

Using the notations: $\varphi = \phi_1 + i\phi_2$, $A_\alpha = (A_1, A_2)$, the covariant derivative of the $U(1)$ connection A_α, $D_\alpha = \partial_\alpha - iA_\alpha$, and its curvature $F_{\alpha\beta} = \partial_{[\alpha}A_{\beta]}$, the self-duality equations (7) are expressed as

$$D_\alpha\varphi = -\varepsilon_{\alpha\beta}[D_\beta(\sin^2\frac{f}{2}\,\mathrm{e}^{ia} + \cos^2\frac{f}{2}\,\mathrm{e}^{ib}) + (\sin^2\frac{f}{2}\,\partial_\beta a + \cos^2\frac{f}{2}\,\partial_\beta b - A_\beta)] \quad (35)$$

$$F_{\alpha\beta} = \frac{1}{s^2}\varepsilon_{\alpha\beta}\{(1+|\varphi|^2) - i[\sin^2\frac{f}{2}(\varphi\mathrm{e}^{-ia} - \varphi^*\mathrm{e}^{-ia}) + \cos^2\frac{f}{2}(\varphi\mathrm{e}^{-ib} - \varphi^*\mathrm{e}^{-ib})]\} \quad (36)$$

$$\varepsilon_{\alpha\beta}\,\partial_\alpha f\,\partial_\beta(a-b) = \frac{2}{s^2\sin f}(1-|\varphi|^2). \quad (37)$$

(35) consists of two distinct complex valued equations, while (36) and (37) consist of one real equation each. Thus we have *six* equations to determine *seven* functions A_α, φ, a, b, and f. Clearly this system is **not** overdetermined, which is what we set out to show here.

We will not solve equations (35), (36) and (37), but will finish with the following observation: Since there are seven functions to be determined by six equations, the system is underdetermined and hence there must be a one $y_\alpha = (s,t)$ dependent parameter family of solutions corresponding to a $U(1)$ gauge freedom. This gauge freedom can be identified by inspection of (35), (36) and (37), and is seen to correspond to

$$\varphi \to \mathrm{e}^{i\Lambda},\ \ A_\alpha \to A_\alpha + \partial_\alpha\Lambda$$

$$a \to a + \Lambda,\ b \to b + \Lambda,\ \ f \to f, \quad (38)$$

which means that removing this gauge arbitrariness, for example as in Ref.[10], we are left with *six* real equations determining *six* real functions.

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